Blocking sets of tangent lines to a hyperbolic quadric in PG(3, 3)

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Abstract

Let $Q^+(3, q)$ be a hyperbolic quadric in $\mathrm{PG}(3, q)$ and $\mathcal{T}$ be the set of all lines of $\mathrm{PG}(3, q)$ which are tangent to $Q^+(3, q)$. If k is the minimum size of a $\mathcal{T}$-blocking set in $\mathrm{PG}(3, q)$, then we prove that $q^2 + 1 \leq k \leq q^2 + q$. When $q = 3$, we show that: (i) there is no $\mathcal{T}$-blocking set of size 10, and (ii) there are exactly two $\mathcal{T}$-blocking sets of size 11 up to isomorphism. By means of the computer algebra systems GAP [13] and Sage [9], we find that there exist no $\mathcal{T}$-blocking sets of size $q^2 + 1$ for each odd prime power $q \leq 13$.

Keywords: Projective space, Blocking set, Conic, Ovoid, Hyperbolic quadric

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1 Introduction

Throughout, q is a prime power. Let $\mathrm{PG}(3, q)$ be the three dimensional projective space defined over a finite field of order q and $Q^+(3, q)$ be a hyperbolic quadric in $\mathrm{PG}(3, q)$. One can refer to [6] for the basic properties of the points, lines and planes of $\mathrm{PG}(3, q)$ with respect to the quadric $Q^+(3, q)$. Every line of $\mathrm{PG}(3, q)$ meets $Q^+(3, q)$ in 0, 1, 2 or $q + 1$ points. We denote by $\mathcal{E}$ (respectively, $\mathcal{T}_1$, $\mathcal{S}$, $\mathcal{T}_0$) the set of lines of $\mathrm{PG}(3, q)$ that intersect $Q^+(3, q)$ in 0 (respectively, 1, 2, $q+1$) points. The elements of $\mathcal{E}$ are called *external lines*, those of $\mathcal{S}$ *secant lines* and those of $\mathcal{T} := \mathcal{T}_0 \cup \mathcal{T}_1$ *tangent lines*. If $L \in \mathcal{T}_i$ with $i \in \{0, 1\}$, then L is also called a $\mathcal{T}_i$-line. The $\mathcal{T}_0$-lines are precisely the lines contained in $Q^+(3, q)$, and so we have $|\mathcal{T}_0| = 2(q+1)$. As every point of $Q^+(3, q)$ is contained in $q-1$ $\mathcal{T}_1$-lines, we have $|\mathcal{T}_1| = (q+1)^2(q-1)$ and hence $|\mathcal{T}| = (q+1)(q^2+1)$. We also have $|\mathcal{S}| = \frac{1}{2}q^2(q+1)^2$ and $|\mathcal{E}| = (q^2+1)(q^2+q+1) - (q+1)(q^2+1) - \frac{1}{2}q^2(q+1)^2 = \frac{1}{2}q^2(q-1)^2$.

For a given nonempty set $\mathcal{L}$ of lines of $\mathrm{PG}(3, q)$, a set X of points of $\mathrm{PG}(3, q)$ is called an *$\mathcal{L}$-blocking set* if each line of $\mathcal{L}$ meets X. The first step in the study of blocking sets has been to determine the smallest cardinality of a blocking set and to characterize, if possible, all blocking sets of that cardinality. If $\mathcal{L}$ is the set of all lines of $\mathrm{PG}(3, q)$ and X

is an $\mathcal{L}$-blocking set, then $|X| \geq q^2 + q + 1$ and equality holds if and only if X is a plane of $\mathrm{PG}(3, q)$. This follows from a more general result by Bose and Burton [4, Theorem 1]. Biondi et al. characterized the minimum size $\mathcal{E}$-blocking sets in [2, Theorem 2.4] for $q \geq 9$ odd and in [1, Theorem 1.1] for $q \geq 8$ even (also see [10, Section 3] for a different proof which works for all even q). When $q > 2$ is even, the minimum size $(\mathcal{E} \cup \mathcal{S})$-blocking sets were determined in [12, Theorem 1.3] using the properties of generalized quadrangles. For $\mathcal{L} \in \{\mathcal{S}, \mathcal{T} \cup \mathcal{S}, \mathcal{E} \cup \mathcal{S}\}$, the minimum size $\mathcal{L}$-blocking sets are described in [11] for all q. When q is even, the minimum size $(\mathcal{E} \cup \mathcal{T})$-blocking sets are characterized in [10, Proposition 1.5].

Suppose q is even and let ζ denote the symplectic polarity of $\mathrm{PG}(3, q)$ associated with the quadric $Q^+(3, q)$. With the symplectic polarity ζ, there is associated a symplectic generalized quadrangle $W(q)$, whose points are the points of $\mathrm{PG}(3, q)$ and whose lines are the lines of $\mathrm{PG}(3, q)$ that are totally isotropic with respect to ζ, with incidence being containment (see [8] for more on generalized quadrangles). The lines of $W(q)$ are precisely the elements of $\mathcal{T}$. If X is a $\mathcal{T}$-blocking set in $\mathrm{PG}(3, q)$, then $|X| \geq q^2 + 1$ and equality holds if and only if X is an ovoid[1] of $W(q)$. There are two families of ovoids known, namely the classical ovoids (being elliptic quadrics of the ambient projective space $\mathrm{PG}(3, q)$) and the Ree-Tits ovoids (which exist only when $q > 2$ is a nonsquare).

In the q odd case, nothing seemed to be known for the minimum size $\mathcal{T}$-blocking sets. If k is the minimum size of such a blocking set, then the following bounds hold by Lemmas 2.1 and 2.2 in the next section:

$$q^2 + 1 \leq k \leq q^2 + q.$$

Calling two $\mathcal{T}$-blocking sets X_1 and X_2 *isomorphic* if there is an automorphism of $\mathrm{PG}(3, q)$ stabilizing $Q^+(3, q)$ and mapping X_1 to X_2, we prove the following (without the aid of a computer) for the case $q = 3$.

Theorem 1.1. *Suppose that $q = 3$. Then there is no $\mathcal{T}$-blocking set of size 10 in $\mathrm{PG}(3, 3)$. Up to isomorphism, there are two $\mathcal{T}$-blocking sets of size 11 in $\mathrm{PG}(3, 3)$.*

In Lemma 2.1 of the next section, we show that a $\mathcal{T}$-blocking set of size $q^2 + 1$ is an ovoid of the subgeometry of $\mathrm{PG}(3, q)$ defined by the tangent lines. In Section 4 of [3], computer code in Sage [9] can be found for classifying ovoids of point-line geometries. With the aid of this code and some computations in GAP [13], we were able to show the nonexistence of $\mathcal{T}$-blocking sets of size $q^2 + 1$ for certain small values of q, see [5].

Theorem 1.2. *There exist no $\mathcal{T}$-blocking sets of size $q^2 + 1$ in $\mathrm{PG}(3, q)$ for each odd prime power $q \leq 13$.*

In Section 2, we prove a few basic results. In Section 3, we construct two nonisomorphic $\mathcal{T}$-blocking sets in $\mathrm{PG}(3, 3)$ each of size 11. Finally, in Section 4, we prove the nonexistence of $\mathcal{T}$-blocking sets of size 10 and classify the $\mathcal{T}$-blocking sets of size 11 in $\mathrm{PG}(3, 3)$.

[1] An *ovoid* of a point-line geometry is a set of points meeting each line in a singleton.

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2 Preliminaries

As in Section 1, consider a hyperbolic quadric $Q^+(3,q)$ in $\mathrm{PG}(3,q)$. A lower bound for the sizes of $\mathcal{T}$-blocking sets is easily derived from the fact that there are $(q+1)(q^2+1)$ tangent lines in total and $q+1$ tangent lines through a given point.

Lemma 2.1. *Let X be a $\mathcal{T}$-blocking set in $\mathrm{PG}(3,q)$. Then $|X| \geq q^2+1$, with equality if and only if every tangent line contains a unique point of X.*

Proof. Each of the $(q+1)(q^2+1)$ tangent lines contains at least one point of X. As every point of $\mathrm{PG}(3,q)$ is contained in precisely $q+1$ tangent lines, we have $|X| \geq \frac{(q+1)(q^2+1)\cdot 1}{q+1} = q^2+1$. Equality holds if and only if every tangent line contains a unique point of X. $\square$

With the quadric $Q^+(3,q)$, there is naturally associated a polarity ζ which is symplectic if q is even and orthogonal if q is odd. For every point x of $Q^+(3,q)$, x^ζ is a plane which is *tangent to $Q^+(3,q)$ at the point x* and intersects $Q^+(3,q)$ in the union of two lines through x. The $q+1$ tangent lines through x are precisely the lines through x contained in x^ζ. By the following lemma, the size of a $\mathcal{T}$-blocking set is bounded above by q^2+q.

Lemma 2.2. *Let π be a plane of $\mathrm{PG}(3,q)$ which is tangent to $Q^+(3,q)$ at the point x. Then $\pi \setminus \{x\}$ is a $\mathcal{T}$-blocking set of size q^2+q.*

Proof. We have $|\pi \setminus \{x\}| = q^2+q$. As every line meets π, every tangent line not containing x meets $\pi \setminus \{x\}$. If L is a tangent line containing x, then L is contained in $x^\zeta = \pi$ and hence contains points of $\pi \setminus \{x\}$. So, $\pi \setminus \{x\}$ is a $\mathcal{T}$-blocking set. $\square$

Suppose q is odd. For every point x of $\mathrm{PG}(3,q) \setminus Q^+(3,q)$, x^ζ is a nontangent plane with $x \notin x^\zeta$ and the set $O_x := x^\zeta \cap Q^+(3,q)$ is a conic of x^ζ. The $q+1$ tangent lines through x are precisely the lines through x meeting O_x. The conic O_x is an *ovoid* of $Q^+(3,q)$, that is, a set of points intersecting each $\mathcal{T}_0$-line in a unique point. The map $x \mapsto O_x$ defines a bijection between $\mathrm{PG}(3,q) \setminus Q^+(3,q)$ and the set of conics contained in $Q^+(3,q)$. When $q=3$, we note that the set of conics contained in $Q^+(3,3)$ coincides with the set of ovoids of $Q^+(3,3)$. If $x \in \mathrm{PG}(3,q) \setminus Q^+(3,q)$, then the number of secant lines through x is equal to $\frac{|Q^+(3,q)\setminus O_x|}{2} = \frac{(q+1)q}{2}$ and the number of external lines through x is equal to $(q^2+q+1)-(q+1)-\frac{(q+1)q}{2} = \frac{(q-1)q}{2}$.

Since q is odd, every point of $x^\zeta \setminus O_x$ lies on 0 or 2 $\mathcal{T}_1$-lines contained in x^ζ. Such a point is called *interior* to O_x in the first case and *exterior* to O_x in the latter. There are $q(q-1)/2$ interior points and $q(q+1)/2$ exterior points in x^ζ with respect to O_x. Every interior point lies on $(q+1)/2$ external lines and $(q+1)/2$ secant lines contained in x^ζ. Every exterior point lies on $(q-1)/2$ external lines and $(q-1)/2$ secant lines contained in x^ζ. Every external line contained in x^ζ contains $(q+1)/2$ interior points and $(q+1)/2$ exterior points. Every secant line contained in x^ζ contains $(q-1)/2$ interior points and $(q-1)/2$ exterior points. One can refer to [7] for these basic properties.

Lemma 2.3. *Suppose $x \in \mathrm{PG}(3,q) \setminus Q^+(3,q)$ with q odd. Then each line of $\mathrm{PG}(3,q)$ through x, which is external to $Q^+(3,q)$, meets x^ζ in a point interior to O_x.*

Proof. Let L be an external line through x. Since $x \notin x^\zeta$, L contains exactly one point of x^ζ. Denote this point by z. We show that z is interior to O_x.

Suppose this is not true. Then z is exterior to O_x. Let M be a $\mathcal{T}_1$-line through z in x^ζ and π be the plane generated by L and M. Then π is a nontangent plane, as it contains the external line L. On the other hand, if y is the unique point of the intersection $M \cap O_x$, then the $\mathcal{T}_1$-line $M_1 := xy$ is contained in π. So π is also the plane generated by the tangent lines M and M_1. It follows that π is the plane which is tangent to $Q^+(3,q)$ at the point y, a contradiction. $\square$

Again under the assumption that $x \in \mathrm{PG}(3,q) \setminus Q^+(3,q)$ with q odd, we denote by $\mathcal{E}_x$ the set of lines in $\mathrm{PG}(3,q)$ through x that are external to $Q^+(3,q)$, and by I_x the set of interior points in x^ζ with respect to the conic O_x. We have $|\mathcal{E}_x| = q(q-1)/2 = |I_x|$. As a consequence of Lemma 2.3, we have the following.

Corollary 2.4. *Suppose $x \in \mathrm{PG}(3,q) \setminus Q^+(3,q)$ with q odd. Then the map from $\mathcal{E}_x$ to I_x, sending each line in $\mathcal{E}_x$ to its point of intersection with I_x, is bijective.*

Proof. By Lemma 2.3, the map is well-defined and is injective. Since $|\mathcal{E}_x| = |I_x|$, the map is surjective also. $\square$

In the special case that $q = 3$, the following can be said.

Lemma 2.5. *Suppose $q = 3$. Let π_1 be a nontangent plane and O_1 be the conic $\pi_1 \cap Q^+(3,3)$ in π_1. Fix a line L in π_1 which is external to O_1. Then there exists exactly one more nontangent plane π_2 satisfying the following:*

1. *L is an external line in π_2 with respect to the conic $O_2 := \pi_2 \cap Q^+(3,3)$.*
2. *If $a \in L$ is exterior (respectively, interior) to O_1 in π_1, then it is also exterior (respectively, interior) to O_2 in π_2.*

In fact, if $a \in L$ is exterior to O_1 in π_1, then the two $\mathcal{T}_1$-lines through a not in π_1 are contained in π_2.

Proof. Let x be the point in $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ such that $O_x = O_1$. Such a point x exists, since the map $\alpha \mapsto O_\alpha := \alpha^\zeta \cap Q^+(3,3)$ is a bijection between $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ and the set of conics contained in $Q^+(3,3)$. We have $\pi_1 = x^\zeta$. Write $L = \{a, b, z_1, z_2\}$, where a, b (respectively, z_1, z_2) are exterior (respectively, interior) to O_1 in π_1. By Corollary 2.4, the lines $T_1 := xz_1$ and $T_2 := xz_2$ are external lines.

Let π_2 be the plane generated by the line L and the point x. Then π_2 is a nontangent plane in which L is external to the conic $O_2 := \pi_2 \cap Q^+(3,3)$. The lines T_1 and T_2 in π_2

are external to O_2. Thus, for $i \in \{1,2\}$, L and T_i are two external lines in π_2 through z_i. It follows that both z_1 and z_2 are interior to O_2 in π_2. This implies that both a and b must be exterior to O_2 in π_2. Hence π_2 satisfies the conditions (1) and (2).

Out of the four $\mathcal{T}_1$-lines through a (respectively, through b), two are contained in π_1 and the other two are in π_2 (as $\pi_1 \cap \pi_2 = L$ is not a $\mathcal{T}_1$-line). This must hold for any nontangent plane satisfying the conditions (1) and (2). This fact implies the uniqueness of π_2 satisfying (1) and (2). □

3 Two constructions of $\mathcal{T}$-blocking sets

In this section, we construct two nonisomorphic $\mathcal{T}$-blocking sets of size 11 each in $\mathrm{PG}(3,3)$.

3.1 First construction

Consider a point $x \in \mathrm{PG}(3,3) \setminus Q^+(3,3)$ and let $I_x = \{z_1, z_2, z_3\}$. Fix a line L in the plane x^ζ which is external to O_x. Then L contains exactly two points of I_x, say z_2 and z_3. Let $\overline{L}$ be the unique line in $\mathcal{E}_x$ such that $\overline{L}$ meets x^ζ in z_1, see Corollary 2.4. Define the following set:

$$B_1 := O_x \cup L \cup \Big(\overline{L} \setminus \{x\}\Big).$$

We prove the following.

Proposition 3.1. *B_1 is a $\mathcal{T}$-blocking set of size* 11 *in* $\mathrm{PG}(3,3)$.

Proof. Clearly, $|B_1| = 11$. Let $A = x^\zeta \setminus B_1$. Then A consists of four exterior points, each of which is different from the two exterior points contained in L. Since every tangent line meets x^ζ, it is enough to prove that each $\mathcal{T}_1$-line through a point of A meets B_1.

Take a point $a \in A$ and a $\mathcal{T}_1$-line T through a. If T is contained in x^ζ, then observe that T meets B_1 in two points, one from O_x and the other one is an exterior point contained in L. So assume that T is not contained in x^ζ. We show that T contains a point of $B_1 \setminus x^\zeta = \overline{L} \setminus \{x, z_1\}$.

Let M be the line in x^ζ through a and z_1. Then M is either external or secant to O_x in x^ζ, as it contains the interior point z_1. Since M has to intersect the external line L in x^ζ in a point different from a and z_1, it follows that M can not be secant to O_x. So M is external to O_x in x^ζ and hence contains an interior point different from z_1. Without loss, we may assume that M contains z_2 as the second interior point.

Setting $\pi_1 = x^\zeta$ and taking the external line M of π_1 in Lemma 2.5, we get a nontangent plane π_2 containing M such that z_1, z_2 are interior points and a is an exterior point in π_2 with respect to the conic $O_2 := \pi_2 \cap Q^+(3,3)$. Note that T is a $\mathcal{T}_1$-line through a in π_2.

Let $\overline{M}$ ($\neq M$) be the second line in π_2 through z_1 which is external to O_2. Out of the three lines through z_1 external to $Q^+(3,3)$, the line M is common to both the planes $\pi_1 = x^\zeta$ and π_2. The plane x^ζ contains one more external line through z_1. So $\overline{M}$ must be the external line through x which corresponds to the point z_1 under the map defined in Corollary 2.4. It follows that $\overline{M} = \overline{L}$. As xz_1 and xz_2 are external lines in π_2 (by

Corollary 2.4), x must be interior to O_2 in π_2. Since the $\mathcal{T}_1$-line T and the external line $\overline{L}$ in π_2 meet in a point exterior to O_2, it follows that T contains a point of $\overline{L} \setminus \{x, z_1\}$. This completes the proof. $\square$

3.2 Second construction

Fix a point $x \in \mathrm{PG}(3,3) \setminus Q^+(3,3)$ and let $I_x = \{z_1, z_2, z_3\}$. Let y be a point in x^ζ exterior to O_x. Let L_1 and L_2 be the two $\mathcal{T}_1$-lines through y which are not contained in x^ζ. For $i \in \{1, 2\}$, let w_i be the tangency point of L_i in $Q^+(3,3)$. Define the following set:

$$B_2 := O_x \cup I_x \cup \Big(L_1 \setminus \{y, w_1\}\Big) \cup \Big(L_2 \setminus \{y, w_2\}\Big).$$

We prove the following:

Proposition 3.2. *B_2 is a $\mathcal{T}$-blocking set of size 11 in* $\mathrm{PG}(3,3)$.

Proof. Clearly, $|B_2| = 11$. Let $D = x^\zeta \setminus B_2$. Then D consists of the six exterior points in x^ζ with respect to O_x. Since every tangent line meets x^ζ, it is enough to prove that each $\mathcal{T}_1$-line through a point of D meets B_2.

Take a point $a \in D$ and a $\mathcal{T}_1$-line T through a. If T is contained in x^ζ, then T meets B_2 in the unique point of $T \cap O_x$. So assume that T is not contained in x^ζ. If $a = y$, then T is either L_1 or L_2 and hence meets B_2 at two points. Assume that $a \neq y$. Since both a and y are exterior to O_x, the line $M := ay$ in x^ζ is either tangent or external to O_x.

Case I: M is tangent to O_x. Let π be the nontangent plane generated by the lines T and M. Denote by O_π the conic $\pi \cap Q^+(3,3)$ in π. The point y in π is exterior to O_π. So there exists one more $\mathcal{T}_1$-line in π (different from M) containing y. Since $\pi \cap x^\zeta = M$, it follows that either L_1 or L_2 is a line in π. Without loss, we may assume that L_1 is a line in π. The lines T and L_1 intersect in π in a point different from y and w_1. So T meets B_2 at a point of $L_1 \setminus \{y, w_1\}$.

Case II: M is external to O_x. Setting $\pi_1 = x^\zeta$ and taking the external line M of π_1 in Lemma 2.5, we get a nontangent plane π_2 through M containing the lines T, L_1 and L_2. Now it can be seen that T intersects L_1 (respectively, L_2) in π_2 at a point different from y and w_1 (respectively, w_2). So T meets B_2 at two points, one from $L_1 \setminus \{y, w_1\}$ and one from $L_2 \setminus \{y, w_2\}$.

Thus B_2 is a $\mathcal{T}$-blocking set of $\mathrm{PG}(3,3)$ of size 11. This completes the proof. $\square$

3.3 The blocking sets B_1 and B_2 are nonisomorphic

Proposition 3.3. *The two blocking sets B_1 and B_2 are nonisomorphic.*

Proof. Write B_2 as a disjoint union $B_2 = (B_2 \cap x^\zeta) \cup (B_2 \setminus x^\zeta)$. Observe that any line meets $B_2 \setminus x^\zeta$ in at most two points. Let R be a line external to $Q^+(3,3)$. If R is a line in x^ζ, then R meets B_2 at exactly two points of $B_2 \cap x^\zeta$ (which come from I_x) and is disjoint from $B_2 \setminus x^\zeta$. Suppose that R is not a line in x^ζ. Then R contains at most one point from

$B_2 \cap x^\zeta$ and at most two points from $B_2 \setminus x^\zeta$. So R is not contained in B_2. Thus every external line meets B_2 in at most three points.

However, from the construction of B_1, it is clear that B_1 contains a line external to $Q^+(3,3)$. So B_1 and B_2 are nonisomorphic. $\square$

4 $\mathcal{T}$-blocking sets of sizes 10 and 11 in $\mathrm{PG}(3,3)$

Consider a hyperbolic quadric $Q^+(3,3)$ in $\mathrm{PG}(3,3)$. We label the points of $Q^+(3,3)$ by x_{ij} where $i,j \in \{1,2,3,4\}$ such that two distinct points x_{ij} and $x_{i'j'}$ of $Q^+(3,3)$ are incident with a $\mathcal{T}_0$-line if either $i = i'$ or $j = j'$.

We denote by O^* the ovoid $\{x_{11}, x_{22}, x_{33}, x_{44}\}$ of $Q^+(3,3)$. There are nine ovoids of $Q^+(3,3)$ that are disjoint from O^*. These are:

$$\begin{aligned}
O_1 &= \{x_{12}, x_{21}, x_{34}, x_{43}\},\ O_2 = \{x_{13}, x_{31}, x_{24}, x_{42}\},\ O_3 = \{x_{14}, x_{41}, x_{23}, x_{32}\},\\
O_4 &= \{x_{12}, x_{24}, x_{43}, x_{31}\},\ O_5 = \{x_{12}, x_{23}, x_{34}, x_{41}\},\ O_6 = \{x_{13}, x_{24}, x_{32}, x_{41}\},\\
O_7 &= \{x_{13}, x_{21}, x_{34}, x_{42}\},\ O_8 = \{x_{14}, x_{21}, x_{32}, x_{43}\},\ O_9 = \{x_{14}, x_{23}, x_{31}, x_{42}\}.
\end{aligned}$$

Lemma 4.1. *There are four collections, each of six ovoids from $\{O_1, O_2, \ldots, O_9\}$, such that every point of $Q^+(3,3) \setminus O^*$ is contained in precisely two ovoids of a given collection. These four collections are $\mathcal{C}^* = \{O_4, O_5, O_6, O_7, O_8, O_9\}$, $\{O_1, O_2, O_5, O_6, O_8, O_9\}$, $\{O_1, O_3, O_4, O_6, O_7, O_9\}$ and $\{O_2, O_3, O_4, O_5, O_7, O_8\}$.*

Proof. It is easily verified that each of these four collections satisfies the required condition. Conversely, suppose that $\mathcal{C} \neq \mathcal{C}^*$ is a collection of six ovoids satisfying the condition of the lemma. As $\mathcal{C} \neq \mathcal{C}^*$, at least one of O_1, O_2, O_3 is contained in $\mathcal{C}$. Now, any partition of $Q^+(3,3) \setminus O^*$ in three ovoids must contain either one or three ovoids of the set $\{O_1, O_2, O_3\}$, implying that at least one of O_1, O_2, O_3 is not contained in $\mathcal{C}$.

Suppose $O_1 \in \mathcal{C}$ and $O_2 \notin \mathcal{C}$. As each of x_{13}, x_{31} should be contained in two ovoids of $\mathcal{C}$, we then must have $O_4, O_6, O_7, O_9 \in \mathcal{C}$. At this stage, x_{12} and x_{21} are already contained in two ovoids of the collection $\mathcal{C}$, implying that O_5 and O_8 do not belong to $\mathcal{C}$. So, $\mathcal{C}$ is necessarily equal to $\{O_1, O_3, O_4, O_6, O_7, O_9\}$.

By symmetry we then see that $\mathcal{C}$ always contains precisely two ovoids of the set $\{O_1, O_2, O_3\}$. If $O_1, O_2 \in \mathcal{C}$ and $O_3 \notin \mathcal{C}$, then a similar reasoning as above shows that $\mathcal{C} = \{O_1, O_2, O_5, O_6, O_8, O_9\}$. Similarly, if $O_2, O_3 \in \mathcal{C}$ and $O_1 \notin \mathcal{C}$, then $\mathcal{C} = \{O_2, O_3, O_4, O_5, O_7, O_8\}$. $\square$

Invoking Lemma 4.1, the verification of the following lemma is straightforward.

Lemma 4.2. *Suppose $\mathcal{C}$ is a collection of six ovoids from $\{O_1, O_2, \ldots, O_9\}$ such that every point of $Q^+(3,3) \setminus O^*$ is contained in precisely two ovoids of $\mathcal{C}$. Let S denote the set of all points $x \in Q^+(3,3) \setminus O^*$ such that $\{x\}$ is the intersection of two distinct ovoids of $\mathcal{C}$. Then $S = Q^+(3,3) \setminus O^*$ if $\mathcal{C} = \mathcal{C}^*$, and $S = O$ if $\mathcal{C} \neq \mathcal{C}^*$, where O is the unique element of $\{O_1, O_2, O_3\}$ not contained in $\mathcal{C}$.*

Lemma 4.3. *Let x be a point of $Q^+(3,3)$ and let $L_1 = \{x, y_1, y_2, y_3\}$ and $L_2 = \{x, z_1, z_2, z_3\}$ be the two $\mathcal{T}_1$-lines through x. Then the following hold:*

(1) *$\{O_{y_1}, O_{y_2}, O_{y_3}\}$ (resp. $\{O_{z_1}, O_{z_2}, O_{z_3}\}$) is a set of ovoids of $Q^+(3,3)$ through x partitioning the set of points of $Q^+(3,3)$ noncollinear with x.*

(2) *If $i, j \in \{1,2,3\}$, then $O_{y_i} \cap O_{z_j}$ contains precisely two points (one of which is x).*

Proof. (1) As L_1 is a $\mathcal{T}_1$-line, we see that $x \in O_{y_i}$ for every $i \in \{1,2,3\}$. Now, take an arbitrary point $u \in Q^+(3,3)$ noncollinear with x. Then u^ζ does not contain x and so intersects L_1 in a unique point y_i. The point y_i is the unique point v of $L_1 \setminus \{x\}$ for which $u \in v^\zeta$. So, $\{O_{y_1}, O_{y_2}, O_{y_3}\}$ partitions the set of points of $Q^+(3,3)$ noncollinear with x. A similar argument holds for the line L_2.

(2) There are six ovoids through the point x. One coincides with O_{y_i}, two (O_{y_r}, O_{y_s}) intersect O_{y_i} in $\{x\}$ where $\{i,r,s\} = \{1,2,3\}$, and the remaining three (necessarily O_{z_1}, O_{z_2}, O_{z_3}) intersect O_{y_i} in two points (one of which is x). □

4.1 Nonexistence of $\mathcal{T}$-blocking sets of size 10

The following result proves the nonexistence of $\mathcal{T}$-blocking sets of size 10 in $\mathrm{PG}(3,3)$.

Proposition 4.4. *There are no $\mathcal{T}$-blocking sets of size 10 in $\mathrm{PG}(3,3)$.*

Proof. Suppose X is a $\mathcal{T}$-blocking sets of size 10 in $\mathrm{PG}(3,3)$. By Lemma 2.1, we then know that each tangent line contains a unique point of X. In particular, $O := X \cap Q^+(3,3)$ is an ovoid of $Q^+(3,3)$ and $Y := X \setminus Q^+(3,3)$ is a set of 6 points outside $Q^+(3,3)$ intersecting each $\mathcal{T}_1$-line in a unique point. Without loss of generality, we may suppose that $O = O^*$. We show the following properties for the collection $\mathcal{C} = \{O_y \,|\, y \in Y\}$:

(a) all ovoids of $\mathcal{C}$ are disjoint from O;

(b) any two ovoids of $\mathcal{C}$ cannot intersect in a singleton;

(c) every point of $Q^+(3,3) \setminus O$ is contained in precisely two ovoids of $\mathcal{C}$.

If O_y with $y \in Y$ contains a point $x \in O$, then the tangent line xy would contain two points of $X = O \cup Y$, namely x and y, a contradiction. If $O_{y_1} \cap O_{y_2}$ is a singleton $\{x\}$, where $y_1, y_2 \in Y$ with $y_1 \neq y_2$, then Lemma 4.3 would imply that there is a $\mathcal{T}_1$-line through x containing y_1 and y_2, a contradiction. Finally, every point $x \in Q^+(3,3) \setminus O$ is contained in two $\mathcal{T}_1$-lines, each containing exactly one point of Y, showing that x is contained in precisely two ovoids of $\mathcal{C}$.

By Lemmas 4.1 and 4.2, we however know that there are no collections $\mathcal{C}$ of six ovoids that satisfy the above properties (a), (b) and (c). □

4.2 Classification of the $\mathcal{T}$-blocking sets of size 11

In the rest of the paper, we classify the $\mathcal{T}$-blocking sets of size 11 in $\mathrm{PG}(3,3)$. We show that there are only two such $\mathcal{T}$-blocking sets up to isomorphism, necessarily isomorphic to the blocking sets B_1 and B_2 constructed in Section 3.

Lemma 4.5. *If X is a $\mathcal{T}$-blocking set of size 11 in $\mathrm{PG}(3,3)$, then $|X \setminus Q^+(3,3)| \in \{6,7\}$ and $|X \cap Q^+(3,3)| \in \{4,5\}$.*

Proof. Since $|X \cap Q^+(3,3)| \leq |X| < 12$, there exists a line L in $Q^+(3,3)$ meeting X in either 1 or 2 points. Suppose every line of $Q^+(3,3)$ meets X in 2 points. Then $|X \cap Q^+(3,3)| = 8$. If L is a line of $Q^+(3,3)$ and $L \setminus X = \{a,b\}$, then each of the four $\mathcal{T}_1$-lines meeting $\{a,b\}$ contains at least one point of $X \setminus Q^+(3,3)$. Any collection of four points of $X \setminus Q^+(3,3)$ that arise in this way are mutually distinct, implying that $|X| = |X \cap Q^+(3,3)| + |X \setminus Q^+(3,3)| \geq 8+4 = 12$, which is a contradiction.

Hence, there exists a line L in $Q^+(3,3)$ meeting X in a unique point. If $L \setminus X = \{a,b,c\}$, then there are six $\mathcal{T}_1$-lines meeting $\{a,b,c\}$ and each of these six $\mathcal{T}_1$-lines contains at least one point of $X \setminus Q^+(3,3)$. Any collection of six points of $X \setminus Q^+(3,3)$ that arise in this way are mutually distinct, implying that $|X \setminus Q^+(3,3)| \geq 6$. As $|X \cap Q^+(3,3)| \geq 4$, we thus have that $|X \setminus Q^+(3,3)| \in \{6,7\}$ and $|X \cap Q^+(3,3)| \in \{4,5\}$. $\square$

Proposition 4.6. *If X is a $\mathcal{T}$-blocking set of size 11 in $\mathrm{PG}(3,3)$, then $|X \cap Q^+(3,3)| = 4$ and $|X \setminus Q^+(3,3)| = 7$.*

Proof. Suppose that this is not the case. Then $|X \cap Q^+(3,3)| = 5$ and $|X \setminus Q^+(3,3)| = 6$ by Lemma 4.5. As each $\mathcal{T}_0$-line contains a point of X, there are precisely two $\mathcal{T}_0$-lines L_1 and L_2 that contain exactly two points of X (while every other $\mathcal{T}_0$-line intersects X in a singleton). The lines L_1 and L_2 belong to distinct parallel classes of lines of $Q^+(3,3)$. We distinguish two cases.

Case I. The singleton $L_1 \cap L_2$ is not contained in X. Without loss of generality, we may suppose that $X \cap Q^+(3,3) = \{x_{12}, x_{13}, x_{21}, x_{31}, x_{44}\}$. The reasoning in Lemma 4.5 leading to the inequality $|X \setminus Q^+(3,3)| \geq 6$ shows that if L is a $\mathcal{T}_0$-line meeting X in a singleton, then any $\mathcal{T}_1$-line meeting $L \setminus X$ cannot contain more than one point of X, and any $\mathcal{T}_1$-line meeting $L \cap X$ cannot contain a point of $X \setminus Q^+(3,3)$. As any point of $Q^+(3,3) \setminus \{x_{11}\}$ is contained in a $\mathcal{T}_0$-line intersecting X in a singleton, we thus see from Lemma 4.3 that any two ovoids O_{y_1} and O_{y_2}, where $y_1, y_2 \in X \setminus Q^+(3,3)$, cannot intersect in a singleton distinct from $\{x_{11}\}$. Also, no ovoid O_y with $y \in X \setminus Q^+(3,3)$ can contain a point of $X \cap Q^+(3,3)$. It can be seen that there are exactly six ovoids of $Q^+(3,3)$ disjoint from $X \cap Q^+(3,3)$ and so these ovoids are precisely the six ovoids O_y, where $y \in X \setminus Q^+(3,3)$. But that is impossible as two of these ovoids, namely $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ and $\{x_{14}, x_{23}, x_{32}, x_{41}\}$, intersect in the singleton $\{x_{23}\} \neq \{x_{11}\}$.

Case II. The singleton $L_1 \cap L_2$ is contained in X. Without loss of generality, we may suppose that $X \cap Q^+(3,3) = O^* \cup \{x_{12}\}$. The reasoning in Lemma 4.5 leading to the inequality $|X \setminus Q^+(3,3)| \geq 6$ shows that if L is a $\mathcal{T}_0$-line meeting $Q^+(3,3) \cap X$ in a singleton, then each of the $\mathcal{T}_1$-lines meeting $L \setminus X$ cannot contain more than one point of X. As any point of $Q^+(3,3) \setminus \{x_{12}\}$ is contained in a line of $Q^+(3,3)$ intersecting X in a singleton, we thus see from Lemma 4.3 that any two ovoids O_{y_1} and O_{y_2}, where $y_1, y_2 \in X \setminus Q^+(3,3)$, cannot intersect in a singleton distinct from $\{x_{12}\}$.

Put $\mathcal{C} = \{O_y \,|\, y \in X \setminus Q^+(3,3)\}$. Then $\mathcal{C}$ is a set of six ovoids of $Q^+(3,3)$, no two of which intersect in a singleton distinct from $\{x_{12}\}$. Moreover, each point $x \in Q^+(3,3) \setminus X$ is contained in precisely two $\mathcal{T}_1$-lines and hence in precisely two ovoids of $\mathcal{C}$.

We count the number of pairs (L, x), where L is a $\mathcal{T}_1$-line disjoint from $X \cap Q^+(3,3)$ and $x \in L \cap X$. There are $|Q^+(3,3) \setminus X| \cdot 2 = 22$ possibilities for L, and each such L contains a unique point of X, implying that there are 22 such pairs. On the other hand, there are 6 possibilities for $x \in X \setminus Q^+(3,3)$.

Since $6 \cdot 3 = 18$, there are at least $22 - 18 = 4$ points of $X \setminus Q^+(3,3)$ whose induced ovoids are disjoint from $Q^+(3,3) \cap X$. There are six ovoids of $Q^+(3,3)$ that are disjoint from $X \cap Q^+(3,3)$:

$$A_1 = \{x_{13}, x_{24}, x_{31}, x_{42}\}, \qquad A_2 = \{x_{14}, x_{23}, x_{32}, x_{41}\}, \qquad A_3 = \{x_{13}, x_{21}, x_{34}, x_{42}\},$$

$$A_4 = \{x_{13}, x_{24}, x_{32}, x_{41}\}, \qquad A_5 = \{x_{14}, x_{23}, x_{31}, x_{42}\}, \qquad A_6 = \{x_{14}, x_{21}, x_{32}, x_{43}\}.$$

Among the six ovoids that we have to choose for the set $\mathcal{C}$, at least four come from the collection $\{A_1, A_2, \ldots, A_6\}$. As exactly two of the six ovoids of $\mathcal{C}$ contain x_{13}, at most two of A_1, A_3, A_4 can occur in $\mathcal{C}$. Similarly, by considering the point x_{14}, we see that at most two of A_2, A_5, A_6 can occur in $\mathcal{C}$. We can conclude that precisely two of A_1, A_3, A_4, as well as precisely two of A_2, A_5, A_6 belong to $\mathcal{C}$. As $A_3 \cap A_4$ and $A_5 \cap A_6$ are singletons distinct from $\{x_{12}\}$, the ovoids A_1 and A_2 must belong to $\mathcal{C}$. Then the fact that $A_3 \cap A_5$, $A_3 \cap A_6$ and $A_4 \cap A_6$ are singletons distinct from $\{x_{12}\}$ implies that A_3 and A_6 cannot belong to $\mathcal{C}$. So, $\mathcal{C}$ certainly contains the ovoids A_1, A_2, A_4 and A_5.

We still need to find two additional ovoids for $\mathcal{C}$. As the points x_{21}, x_{34} and x_{43} are not contained in $A_1 \cup A_2 \cup A_4 \cup A_5$ and need to be covered twice, each of these two ovoids should contain these points. But that is impossible as there is only one ovoid containing these three points, namely $\{x_{12}, x_{21}, x_{34}, x_{43}\}$. $\square$

In the sequel, we suppose that X is a set of 11 points of $\mathrm{PG}(3,3)$ that is a $\mathcal{T}$-blocking set. Then $|X \cap Q^+(3,3)| = 4$ and $|X \setminus Q^+(3,3)| = 7$ by Proposition 4.6. In fact, $U_1 := X \cap Q^+(3,3)$ is an ovoid of $Q^+(3,3)$. Denote by U_2 the subset of $Q^+(3,3)$ consisting of the following points:

- points of $X \cap Q^+(3,3)$ contained in a $\mathcal{T}_1$-line that contains points of $X \setminus Q^+(3,3)$,
- points of $Q^+(3,3) \setminus X$ contained in a $\mathcal{T}_1$-line that contains at least two points of $X \setminus Q^+(3,3)$.

Lemma 4.7. *The set U_2 is an ovoid of $Q^+(3,3)$.*

Proof. Let L be a line of $Q^+(3,3)$ and put $\{x_L\} := L \cap U_1$. For every $y \in X \setminus Q^+(3,3)$ denote by y' the unique point of $L \cap O_y$, that is, the unique point y' of L for which yy' is a $\mathcal{T}_1$-line. Each $\mathcal{T}_1$-line meeting $L \setminus \{x_L\}$ contains at least one point of $X \setminus Q^+(3,3)$, and so each point of $L \setminus \{x_L\}$ is the image of at least two points of $X \setminus Q^+(3,3)$ under the map $y \mapsto y'$. So, precisely one of the following two cases occurs:

(a) The point x_L is the image of precisely one point of $X \setminus Q^+(3,3)$ and each of the three points of $L \setminus \{x_L\}$ is the image of precisely two points of $X \setminus Q^+(3,3)$.

(b) There exists a unique point x'_L on $L \setminus \{x_L\}$ which is the image of precisely three points of $X \setminus Q^+(3,3)$, each of the two remaining points of $L \setminus \{x_L\}$ is the image of precisely two points of $X \setminus Q^+(3,3)$. In this case, the point x_L itself is not the image of any point of $X \setminus Q^+(3,3)$.

In case (a), we see that x_L is the unique point of U_2 on L. In case (b), we see that x'_L is the unique point of U_2 on L. Since $L \cap U_2$ is always a singleton, we conclude that U_2 must be an ovoid of $Q^+(3,3)$. □

Now, if $\mathcal{C}$ is the collection of the seven ovoids O_y, where $y \in X \setminus Q^+(3,3)$, then the following properties hold:

(P1) No point of $U_1 \setminus U_2$ is contained in an ovoid of $\mathcal{C}$.

(P2) Every point of $U_1 \cap U_2$ is contained in precisely one ovoid of $\mathcal{C}$.

(P3) Every point of $Q^+(3,3) \setminus (U_1 \cup U_2)$ is contained in precisely two ovoids of $\mathcal{C}$.

(P4) Every point of $U_2 \setminus U_1$ is contained in precisely three ovoids of $\mathcal{C}$.

(P5) No two ovoids of $\mathcal{C}$ intersect in a singleton $\{x\}$, where $x \in Q^+(3,3) \setminus (U_1 \cup U_2)$.

(P6) No three ovoids of $\mathcal{C}$ can mutually intersect in the same singleton $\{x\}$, where $x \in U_2 \setminus U_1$.

Proposition 4.8. *Suppose that U_1 and U_2 are two (not necessarily distinct) ovoids of $Q^+(3,3)$. Let Y be a set of seven points of* $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ *and put $\mathcal{C} := \{O_y \,|\, y \in Y\}$. If $\mathcal{C}$ satisfies the properties (P1) – (P6) above, then $U_1 \cup Y$ is a $\mathcal{T}$-blocking set of size* 11.

Proof. We have $|U_1 \cup Y| = 11$. Since U_1 is an ovoid of $Q^+(3,3)$, every $\mathcal{T}_0$-line meets U_1 at a unique point. Every $\mathcal{T}_1$-line through a point of U_1 obviously meets U_1. By (P4) and (P6), every $\mathcal{T}_1$-line through a point of $U_2 \setminus U_1$ contains a point of Y. By (P3) and (P5), every $\mathcal{T}_1$-line through a point of $Q^+(3,3) \setminus (U_1 \cup U_2)$ contains a point of Y. □

We now use the above result to classify the $\mathcal{T}$-blocking sets of size 11 in $\mathrm{PG}(3,3)$. We assume that U_1 and U_2 are two ovoids of $Q^+(3,3)$ and that $\mathcal{C}$ is a collection of seven ovoids of $Q^+(3,3)$ satisfying the properties (P1) – (P6) above. If Y is the set of seven points of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ for which the collection $\{O_y \,|\, y \in Y\}$ coincides with $\mathcal{C}$, then $X = U_1 \cup Y$ is a $\mathcal{T}$-blocking set of size 11 by Proposition 4.8. Without loss of generality, we may suppose that $U_1 = O^* = \{x_{11}, x_{22}, x_{33}, x_{44}\}$. Then the nine ovoids disjoint from $U_1 = \{x_{11}, x_{22}, x_{33}, x_{44}\}$ are $O_1, O_2, \ldots, O_9$ as defined in the beginning of this section.

The ovoid U_2 can have five positions with respect to U_1 (up to isomorphism):

I: $U_2 = \{x_{11}, x_{22}, x_{33}, x_{44}\} = U_1$,

II: $U_2 = \{x_{11}, x_{22}, x_{34}, x_{43}\}$,

III: $U_2 = \{x_{11}, x_{23}, x_{34}, x_{42}\}$,

IV: $U_2 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$,

V: $U_2 = \{x_{12}, x_{23}, x_{34}, x_{41}\}$.

Treatment of Case I

In this case, (P2) implies that the points of $U_1 \cap U_2 = U_1 = U_2$ are partitioned by certain ovoids of $\mathcal{C}$. The partition has shape 4, $2+2$, $2+1+1$ or $1+1+1+1$, leading to four subcases.

(Ia) Suppose the mentioned partition has shape 4. Then $U_1 = U_2 \in \mathcal{C}$. Again (P2) implies that every ovoid of $\mathcal{C} \setminus \{U_1\}$ is disjoint from $U_1 = U_2$. By (P3), $\mathcal{C} \setminus \{U_1\}$ is a collection of six ovoids as in Lemma 4.1. A contradiction is then readily obtained from Lemma 4.2 and property (P5).

(Ib) Suppose the mentioned partition has shape $2+2$. Without loss of generality, we may suppose that $\{x_{11}, x_{22}, x_{34}, x_{43}\}$ and $\{x_{33}, x_{44}, x_{12}, x_{21}\}$ belong to $\mathcal{C}$. By (P2), each of the remaining five ovoids of $\mathcal{C}$ is disjoint from $U_1 = U_2$. So we need to find five additional ovoids from the collection $\{O_1, O_2, \ldots, O_9\}$. By (P3) and (P5), the second ovoid of $\mathcal{C}$ through x_{12} must contain x_{21} and therefore be equal to $O_1 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$. As x_{12}, x_{21}, x_{34} and x_{43} have already been covered twice, the remaining four ovoids should be contained in $\{x_{13}, x_{14}, x_{23}, x_{24}, x_{31}, x_{32}, x_{41}, x_{42}\}$ and hence equal to O_2, O_3, O_6 and O_9. One readily verifies that the collection consisting of the seven ovoids $\{x_{11}, x_{22}, x_{34}, x_{43}\}$, $\{x_{33}, x_{44}, x_{12}, x_{21}\}$, O_1, O_2, O_3, O_6 and O_9 satisfies the properties (P1) – (P6).

(Ic) Suppose the mentioned partition has shape $2+1+1$. Without loss of generality, we may suppose that $\{x_{11}, x_{22}, x_{34}, x_{43}\}$ is present in $\mathcal{C}$. Then the ovoid $\{x_{12}, x_{21}, x_{33}, x_{44}\}$ is not in $\mathcal{C}$. By (P3) and (P5), the second ovoid of $\mathcal{C}$ through x_{34} must contain x_{43} and hence coincides with $O_1 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$. Note that each of x_{34}, x_{43} has now been covered twice, while each of x_{12} and x_{21} only once. Therefore the second ovoid of $\mathcal{C}$ through x_{12}, which cannot intersect $\{x_{12}, x_{21}, x_{34}, x_{43}\}$ in a singleton, must also contain x_{21}. But that is impossible as the two ovoids through $\{x_{12}, x_{21}\}$, namely O_1 and $\{x_{12}, x_{21}, x_{33}, x_{44}\}$ are already forbidden.

(Id) Suppose the mentioned partition has shape $1+1+1+1$. Without loss of generality, we may suppose that $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ belongs to $\mathcal{C}$. Each $y \in \{x_{23}, x_{34}, x_{42}\}$ is contained in a second ovoid of $\mathcal{C}$ which meets $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ in a second point $y' \in \{x_{23}, x_{34}, x_{42}\}$. But then the pairs $\{y, y'\}$ would partition $\{x_{23}, x_{34}, x_{42}\}$, an obvious contradiction.

Treatment of Case II

We have $U_2 = \{x_{11}, x_{22}, x_{34}, x_{43}\}$. If $U_2 \in \mathcal{C}$, then by (P1) – (P4), $\mathcal{C} \setminus \{U_2\}$ is a collection of six ovoids as in Lemma 4.1. A contradiction is then readily obtained from Lemma 4.2 and property (P5). So, $U_2 \notin \mathcal{C}$. By (P1) and (P2), it follows that the unique ovoid of $\mathcal{C}$ containing x_{11} is either $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ or $\{x_{11}, x_{24}, x_{32}, x_{43}\}$. In view of the symmetry $3 \leftrightarrow 4$, we may without loss of generality suppose that $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ is the unique ovoid of $\mathcal{C}$ containing x_{11}. There are still six ovoids to choose for $\mathcal{C}$, one of them contains x_{22} and the other five are contained in the collection $\{O_1, O_2, \ldots, O_9\}$. None of these six ovoids can intersect $\{x_{11}, x_{23}, x_{34}, x_{42}\}$ in the singleton $\{x_{23}\}$ or the singleton $\{x_{42}\}$, implying that O_2 and O_3 do not belong to $\mathcal{C}$. So, we need to take five ovoids among the seven ovoids $O_1, O_4, O_5, O_6, O_7, O_8, O_9$. Since $O_4 \cap O_5 = \{x_{12}\}$, $O_5 \cap O_6 = \{x_{41}\}$, $O_4 \cap O_6 = \{x_{24}\}$ and $O_7 \cap O_9 = \{x_{42}\}$, (P5) implies that none of the pairs $\{O_4, O_5\}$, $\{O_5, O_6\}$, $\{O_4, O_6\}$, $\{O_7, O_9\}$ can be contained in $\mathcal{C}$. So, two among O_4, O_5, O_6 cannot be in $\mathcal{C}$, as well as one among O_7, O_9. So, it is impossible to find the five required ovoids from the collection $\{O_1, O_4, O_5, \cdots, O_9\}$.

Treatment of Case III

We have $U_2 = \{x_{11}, x_{23}, x_{34}, x_{42}\}$. If $U_2 \in \mathcal{C}$, then by (P1) – (P4), $\mathcal{C} \setminus \{U_2\}$ is a collection of six ovoids as in Lemma 4.1. A contradiction is then readily obtained from Lemma 4.2 and property (P5). So, $U_2 \notin \mathcal{C}$. Then, using (P1) and (P2), the unique ovoid of $\mathcal{C}$ containing x_{11} must be $\{x_{11}, x_{24}, x_{32}, x_{43}\}$. Each point $y \in \{x_{24}, x_{32}, x_{43}\}$ is contained in a second ovoid of the collection $\mathcal{C}$ which meets $\{x_{11}, x_{24}, x_{32}, x_{43}\}$ in a second point $y' \in \{x_{24}, x_{32}, x_{43}\}$. Then the pairs $\{y, y'\}$ would partition $\{x_{24}, x_{32}, x_{43}\}$, an obvious contradiction.

Treatment of Case IV

We have $U_2 = \{x_{12}, x_{21}, x_{34}, x_{43}\}$. By (P1), all ovoids of $\mathcal{C}$ are disjoint from U_1. So we have to choose seven ovoids for $\mathcal{C}$ among the nine ovoids $O_1, O_2, \ldots, O_9$. By (P4), there are three ovoids of $\mathcal{C}$ containing x_{12}. So the ovoids O_1, O_4 and O_5 belong to $\mathcal{C}$. As $O_4 \cap O_6 = \{x_{24}\}$ and $O_4 \cap O_9 = \{x_{31}\}$, the ovoids O_6 and O_9 are not in $\mathcal{C}$ by (P5). Hence, $\mathcal{C} = \{O_1, O_2, O_3, O_4, O_5, O_7, O_8\}$. One readily verifies that this collection of ovoids satisfies the properties (P1) – (P6).

Treatment of Case V

Here $U_2 = \{x_{12}, x_{23}, x_{34}, x_{41}\}$. By (P1), all ovoids of $\mathcal{C}$ are disjoint from U_1. So we have to choose seven ovoids for $\mathcal{C}$ among the nine ovoids $O_1, O_2, \ldots, O_9$. Since $O_4 \cap O_6 = \{x_{24}\}$, $O_4 \cap O_8 = \{x_{43}\}$ and $O_4 \cap O_9 = \{x_{31}\}$, O_4 cannot occur in $\mathcal{C}$ by (P5). Since $O_6 \cap O_7 = \{x_{13}\}$ and $O_6 \cap O_8 = \{x_{32}\}$, we then know that also O_6 cannot occur in $\mathcal{C}$. So, we should have that $\mathcal{C} = \{O_1, O_2, O_3, O_5, O_7, O_8, O_9\}$. But that is impossible again by (P5) as $O_7 \cap O_8 = \{x_{21}\}$.

Let $X_1 = U_1 \cup Y_1 = O^* \cup Y_1$, where Y_1 is the set of seven points of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ for which the collection $\{O_y \,|\, y \in Y_1\}$ consists of the ovoids $\{x_{11}, x_{22}, x_{34}, x_{43}\}$, $\{x_{33}, x_{44}, x_{12}, x_{21}\}$, O_1, O_2, O_3, O_6 and O_9 of $Q^+(3,3)$. Similarly, let $X_2 = U_1 \cup Y_2 = O^* \cup Y_2$, where Y_2 is the set of seven points of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ for which the collection $\{O_y \,|\, y \in Y_2\}$ coincides with $\{O_1, O_2, O_3, O_4, O_5, O_7, O_8\}$. Note that X_1 is associated with the seven ovoids corresponding to subcase (Ib) in the treatment of Case I and X_2 is associated with the seven ovoids in the treatment of Case IV.

By the above discussion, we thus know:

Proposition 4.9. *Up to isomorphism, X_1 and X_2 are the two $\mathcal{T}$-blocking sets of size 11 in $\mathrm{PG}(3,3)$.*

Our intention is now to identify the two blocking sets X_1 and X_2 with that of B_1 and B_2 constructed, respectively, in Sections 3.1 and 3.2. We shall rely on the following lemma.

Lemma 4.10. *Every ovoid O of $Q^+(3,3)$ is contained in precisely four partitions of $Q^+(3,3)$ into ovoids. Three of these are induced by external lines.*

Proof. Without loss of generality, we may suppose that $O = O^*$. The partitions then have the form $\{O^*, O_i, O_j, O_k\}$, where $i, j, k \in \{1, 2, \ldots, 9\}$ with $i < j < k$. It is straightforward to verify that these partitions are $\{O^*, O_1, O_2, O_3\}$, $\{O^*, O_1, O_6, O_9\}$, $\{O^*, O_2, O_5, O_8\}$ and $\{O^*, O_3, O_4, O_7\}$. Now, let x denote the unique point of $\mathrm{PG}(3,3) \setminus Q^+(3,3)$ for which $O_x = O = O^*$. There are three external lines through x. If $\{x, u_1, u_2, u_3\}$, $\{x, u_4, u_5, u_6\}$ and $\{x, u_7, u_8, u_9\}$ are these external lines, then the nine ovoids $\{O_{u_1}, O_{u_2}, \ldots, O_{u_9}\}$ are mutually distinct. So, $\{O^*, O_1, O_6, O_9\}$, $\{O^*, O_2, O_5, O_8\}$ and $\{O^*, O_3, O_4, O_7\}$ must be the partitions among the four that are induced by external lines. $\square$

Proposition 4.11. *There exist two mutually disjoint external lines K, L and a point $x \in K$ such that $X_1 = O_x \cup (K \setminus \{x\}) \cup L$.*

Proof. Let K denote the external line determined by the ovoids O^*, O_1, O_6, O_9, and denote by x the unique point of K for which $O_x = O^*$. Among the four partitions of $Q^+(3,3)$ into ovoids containing O_2, $\{O^*, O_1, O_2, O_3\}$ is not induced by any external line (see the proof of Lemma 4.10). So, again by Lemma 4.10, the partition of $Q^+(3,3)$ by the ovoids $\{x_{11}, x_{22}, x_{34}, x_{43}\}$, $\{x_{33}, x_{44}, x_{12}, x_{21}\}$, O_2, O_3 is induced by some external line, say L. Then we have $K \cap L = \emptyset$ and $X_1 = O_x \cup \Big(K \setminus \{x\}\Big) \cup L$. $\square$

By Proposition 3.3, we know that the two blocking sets B_1 and B_2 constructed in Sections 3.1 and 3.2 are nonisomorphic. In fact, by the proof of that proposition, we know that B_2 does not contain any external line, while B_1 does. Comparing this with Propositions 4.9 and 4.11, we then conclude that the blocking set X_1 is isomorphic to B_1 and that the blocking set X_2 is isomorphic to B_2.

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